Combinatorial Analysis Lecture 3

Combinations

How many subsets of size r can be picked from a set of size n, $n \geq r$?

$$\underbrace{1 \quad 2 \quad 3 \quad \dots \quad r}_{\substack{\text{selected} \\ r}} \quad \underbrace{r+1 \quad \dots \quad n}_{\substack{\text{not selected} \\ n-r}}$$

- $n!$ linear orderings of objects
- rearrangements within the red or blue trays doesn't affect which objects were selected.
- Each "object" selection pattern corresponds to $r!(n-r)!$ permutations.

For example, objects $1, 2, \dots, r$ can be selected via permutation $(1, 2, \dots, r, r+1, \dots, n)$ or $(2, 1, \dots, r, r+1, \dots, n)$ etc.

Thus, the total number of subsets of size r

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

<u>Remark:</u> $\binom{n}{0} = 1$

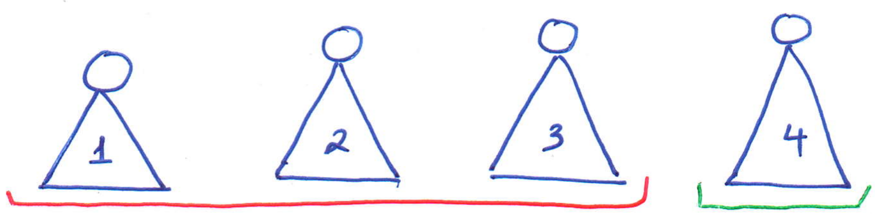
Only one empty subset.

To make the formula work $\binom{n}{0} = \frac{n!}{0!\,(n-0)!}$

$= \frac{1}{0!} = 1$, we define $0! = 1$.

<u>Ex.</u> 3 students are selected from a group of 4 students to go on a field trip to Greenwood. How many choices are possible?

<u>Solution:</u>

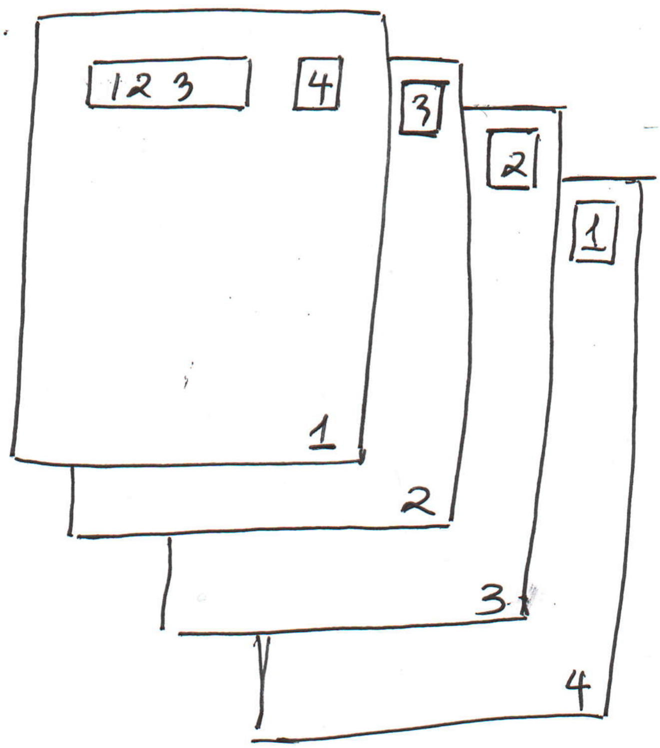


Can you construct a Kafka document, indicating which students were picked?

One possible protocol is to have the individuals listed by their numeric order.

For example, if individuals 1,3,4 were picked list $\boxed{1\,3\,4}$ $\boxed{2}$ but never ~~$\boxed{1\,4\,3}$ $\boxed{2}$~~

We thus have



$$\binom{4}{3} = \frac{4!}{3!\,1!} = 4 \quad \text{documents.}$$

<u>Ex.</u> From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?
What if 2 of the men are feuding and refuse to serve together?

<u>Solution:</u>

Protocol. Index women: 1–5, men: 1–7.

$\boxed{\qquad\qquad\qquad}$ $\qquad$ $\boxed{\qquad\qquad\qquad}$

women (listed by index) $\qquad$ men (listed by index)

For example, if the selected women are 2, 5 and the selected men are 4, 1, 7 we input

$\boxed{2\ \ 5}$ $\qquad$ $\boxed{1\,4\ \ 7}$

By the basic principle of counting, the number of possible codes is $\binom{5}{2}\binom{7}{3}$.

Now, how many outcomes are there, in which the two enemies are selected?

$$\begin{array}{cc} \boxed{\quad\quad\quad} & \boxed{1 \;\; 2 \;\; *} \\ W & M \end{array}$$

Suppose these men are labeled 1 and 2. Clearly we want to count the number of codes where

$$\boxed{* \; *} \qquad \boxed{1 \;\; 2 \;\; *}$$

There are exactly $\binom{5}{2}\binom{7-2}{1} = \binom{5}{2}\cdot 5$ such codes.

Thus, there are a total of $\binom{5}{2}\binom{7}{3} - \binom{5}{2}\cdot 5$ committees without both enemies together.

Alternatively we have

$$\underbrace{\binom{7-2}{3}}_{\text{both enemies absent}} + \underbrace{\binom{7-2}{2}}_{\text{enemy \#1 in}} + \underbrace{\binom{7-2}{2}}_{\text{enemy \#2 in}}$$

$$= \binom{5}{3} + 2\binom{5}{2}$$

ways of selecting 3 men, no two of which are feuding.

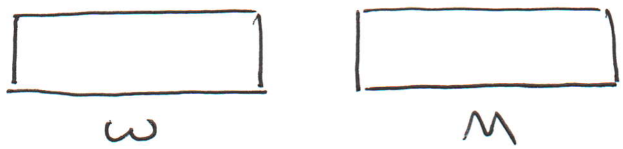
Thus $\binom{7}{3} - \binom{5}{1} = \binom{5}{3} + 2\binom{5}{2}$

and the number of desired committees is

$$\binom{5}{2}\left(\binom{7}{3} - \binom{5}{1}\right) = \binom{5}{2}\left(\binom{5}{3} + 2\binom{5}{2}\right)$$

<u>Ex,</u> 10 women, 16 men. Committee consisting of 7 women and 9 men to be formed. 5 men are friends and will only serve together. 2 women refuse to serve together. How many committees are possible?

Solution:



- lable women 1-10 with 1,2 enemies
- lable men 1-16 with 1-5 friends.
- box W cannot contain 1,2, box M either contains 1-5 or doesn't contain any of the numbers 1-5.

W:
$$\underbrace{\binom{10}{7}}_{\text{All 7 digit codes}} - \underbrace{\binom{10-2}{7-2}}_{\text{codes that begin with 1,2}}$$

M:
$$\underbrace{\binom{16-5}{9}}_{\text{codes without } 1 2 .. 5} + \underbrace{\binom{16-5}{9-5}}_{\text{codes beginning in } 1 2 3 .. 5}$$

Thus the number of possible committees is $W \cdot M = \left[\binom{10}{7} - \binom{8}{5}\right]\left[\binom{11}{4} + \binom{11}{9}\right]$

Ex. n antennas of which m are defective and $n-m$ are functional. Defectives cannot be distinguished among themselves. Functionals cannot be distinguished among themselves.

How many linear orderings are there in which no two defectives are consecutive?

Solution:

$$\underset{1}{1} \quad \underset{2}{1} \quad \dots \quad \underset{n-m}{1} \quad - \text{ functional}$$

$$\underset{1}{0} \quad \underset{2}{0} \quad \dots \quad \underset{m}{0} \quad - \text{ defective}$$

<u>Trick</u>: Don't think about the antenas.

Think about the spaces between them

$$\underset{1}{\wedge}\ 1\ \underset{2}{\wedge}\ 1\ \underset{3}{\wedge}\ \cdots\ \underset{n-m}{\wedge}\ 1\ \underset{n-m+1}{\wedge}$$

Each distinct space can be occupied by at most one defective antena.

Out of $n-m+1$ spaces, we must pick m to be occupied by defective antenas.

Thus $\binom{n-m+1}{m}$ is the number of distinct linear orderings.